Spline Curves for Geometric Modelling of Highway Design

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Abstract: *The new Road Design Italian Standards (“Norme funzionali e geometriche per la costruzione delle strade”, D.M. 5.11.2001), allows the use of polynomial curves for the design of the road alignment design. Such polynomial curves permit to define effectively road alignment regarding the geometric-dynamic constraint. Really, these curves allow the unitary spatial definition of the road alignment in a numerical form, along an entire road section; these representation methods are very suitable for the design of transport infrastructures (roads, railways, levitation systems, cableways, etc.), when the terrain is represented too through of a three-dimensional numerical model (DTM).*
For these curves, it is possible to define the characteristic parameters (degree, continuity, shape, curvature, etc.) on the basis of the control of cinematic and dynamic indexes, regarding the vehicle motion.
In this paper some kinds of polynomial-parametric curves, proper for the design of road alignment, are examined, making also the performance comparisons with traditional design methodologies.
The matrix formulation of these curves is specially appropriate for use in CAD systems and will be implemented in the driving simulation software, currently under development in Roads Department of University of Rome ”La Sapienza”.

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INTRODUCTION

The present new millennium is a time for reminds to us how far we have come in transportation research and how much there is still to be done. History tells us that at some times and in some modes of transportation less than 100 years is needed for a revolution to occur. Many of the broad principles of highway geometric design were formulated in the 1930s and 1940s and have changed today only in their minor details. In the foreseeable future, we expect the changes in transportation infrastructures to be really evolutionary but not revolutionary, due to the fact, for instance, that the maglev system (transportation system without the wheels) should be the system that overpass the wheel era.

However the information revolution will continue as computers, communications, and sensors become increasingly fast and cheap. Highway administrations will have in the future the access to complete data sets on vehicle positions and speeds and road conditions. From these data, it is possible to generate better traffic management tools, better accident prediction models, and more realistic simulations of proposed improvements.

We see a need to codify safety knowledge and provide analytical tools to predict the safety performance of existing roadways and the anticipated safety performance of proposed projects. The accuracy of any safety prediction tools that are developed depends on the reliability of research results concerning the safety effects of geometric features. Major advances in research techniques have recently been made and more are expected.

The current philosophy of design is based on the implicit assumption that any design developed in accordance with established geometric design standard is safe and that any design that does not meet key aspects of established policies is unsafe. Many engineers take this for granted, and the courts appear to base decisions in tort liability cases on this philosophy.

The geometric modelling of a road layout is still currently faced with a methodology strongly consolidated, that is founded on the acquisition of the ground model (numerical, topographical, etc.) and on the consequent definition of the plain and vertical course of the axle, in adherence to the technical-economic ties of the project. Such definition happens through the composition of a lot of adjoining lines constituted by bending elements, constant and to varying bending, with the purpose to form a project line characterized by continuity at least equal to C^2. The usual design procedure foresees that, to a consistent initial phase in the composition of the plain and vertical elements, that it happens in disarticulated way, follow an activity of coordination of the two different and separated projections (plain and vertical) of the layout following geometric, dynamic and optic criteria. The final phase should consist in the global verification of the three-dimensional line of project, produced in adherence to the performance prescriptions and, besides, such verification is developed only generally in formal way.

The non-numerical definition of the geometric elements, and not unitary, it doesn't allow the planning of the road layout as line defined in the 3-D space. It achieves of it that, although there has been a consistent development of the CAD systems, the assisted road planning it rather results currently artificial and it strongly anchors conditioned by the executive methodology.

With the publication in Italy of the *"Norme geometriche e funzionali per la costruzione delle strade"* (D.M. 5.11.2001) it is allowed the use of polynomial lines for the planning of road axes now, favouring so the treatment of the problem in unitary form.

In the present work is developed a series of polynomial type parametric curves (of the Splines family), usable for the modelling in very unitary form of road axles, and the principal problems, the advantages and the emerged disadvantages are discussed.

It is also examined the use to the techniques of simulation of the road use. In fact the particular formulation used for the geometric definition of the layout allows the use of it, without necessity of further elaborations, for numerical-graphics applications, as - for instance - the simulation of the motion of the vehicles on these road layouts.

INTEREST OF THE POLYNOMIAL-PARAMETRIC CURVES

To understand the main advantages of parametric forms it is necessary to consider the difficulty or impossibility to define all the geometric elements through single value ordinary functions, such as $y = f(x)$, in geometric modelling practice. In fact for frame of reference dependence of the functions to single value, for difficult treatment of function's endless or indefinite slope and for different mathematical expressions of geometric entities, use of single value functions is extremely difficult in the analytical representation of road alignment geometry. Nevertheless, parametric form of polynomial functions allows to form road alignment

through a series of points in the space; such possibility derives from the use of a matrix expression with polynomial curves, for which it is possible to impose suitable continuity characteristics to form a road alignment in unitary way. Besides, being the procedure based on design constraint imposition, the generated road alignment verifies immediately kinematic and dynamic parameters of vehicles motion and road design Standards. Parametric forms allows easy treatment through automatic calculation. In fact this mathematical curve expression allows an easy implementation through calculation "routines" to determine road alignment points.

CONSIDERED PARAMETRIC CURVES

In this paper the selection of suitable polynomial-parametric curves used for road design is based on the fulfilment of the road alignment constraints , e.g.:

- feasibility of definition and consequent variation of road alignment in relationship to altimetric and planimetric constraints;
- monotonic road alignment;
- minimum C^2 continuity;
- possibility to impose initial and final road alignment tangents;
- possibility to impose initial and final curvature value;
- possibility of curvature variation along road alignment (in particular limited propagation of local variation effects);
- possibility to use elements with null curvature (suitable in case of derivation from existing road alignment);
- possibility to control geometric and dynamic parameters (curvature and jerk) together the definition of road alignment.

Curve-defining techniques that interpolate and that approximate are used; the first curves pass exactly through control points, the further approximate given points and don't pass through all control points.

Cubic Hermite Polynomial

These curves (Bartels et al., 1987) are the union of *n* $Q_i(u)$ polynomial spatial skew segments interpolating a given series of n+1 spatial points (P_0, P_1, … , P_n). For each segment $Q_i(u)$ $[x_i(u),y_i(u),z_i(u)]$ the following three cubic polynomial expressions can be written:

$$
\begin{aligned}
x_i(u) &= a_{3x,i}u^3 + a_{2x,i}u^2 + a_{1x,i}u + a_{0x,i} \\
y_i(u) &= a_{3y,i}u^3 + a_{2y,i}u^2 + a_{1y,i}u + a_{0y,i} \\
z_i(u) &= a_{3z,i}u^3 + a_{2z,i}u^2 + a_{1z,i}u + a_{0z,i}
\end{aligned}
\qquad (1)
$$

with $u \in [0; 1]$ and i=1,2,…,n.

The effects of the control points movement propagate in all the curve with shape, characteristic and properties modifications. In road design, Hermite curves are defined by the algebraic coefficients of parametric expressions (1), through positioning of the n+1 control points; such positioning comes from design and terrain constraints.

For *n+1* control points, interpolated by *n* cubic polynomial segments, 4·3=12 coefficients are requested to define each segment, therefore:

(2) $4 \cdot 3 \cdot n$

coefficients are to be determined (4·n coefficients for each x-u, y-u, z-u plane).

The available equations (for each plane) are the following:

- 2·n equations for segment passing through control points;
- (n-1) equations for first derivative continuity in contact points (C^1 continuity);
- (n-1) equations for second derivative continuity in contact points (C^2 continuity).

The total equation count is:

(3) $4 \cdot n - 2$

while the needed equations is 4n.

We obtain the two further equations setting tangent and curvature value at the beginning and at the end of the curve, through first and second parametric derivative value. The system of equations can be expressed through control points coordinate, first and second derivative of internal and external control points.

In this way, (Figure 1) for the i^{th} polynomial segment of the curve in y-u plane, the conditions for the final point are:

(4) $Y_i(0) = a_{0y,i} = y_{i-1}$

$Y_i(1) = a_{3y,i} + a_{2y,i} + a_{1y,i} + a_{0y,i} = y_i$

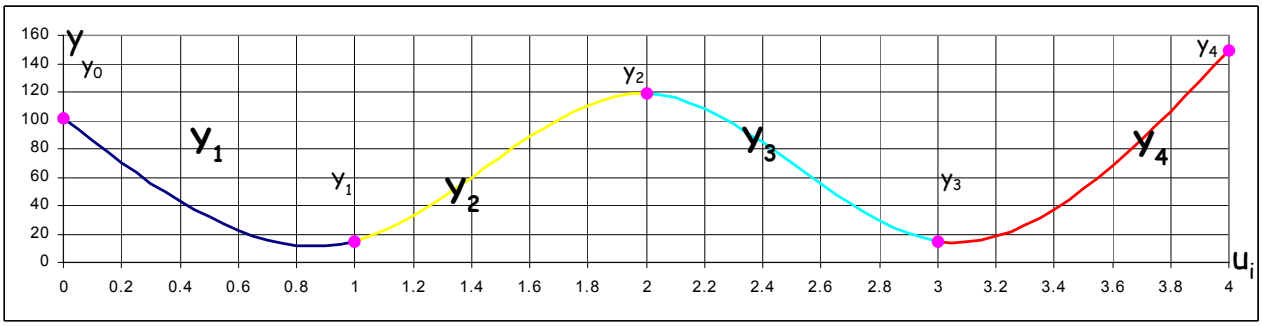


Figure 1

and for the first derivative at the beginning ($D_{y,i-1}$) and at the end ($D_{y,i}$) of i^{th} polynomial segments:

(5) $Y_i'(0) = a_{1y,i} = D_{y,i-1}$
$Y_i'(1) = 3 \cdot a_{3y,i} + 2 \cdot a_{2y,i} + a_{1y,i} = D_{y,i}$

In the equations (5), y_{i-1}, y_i, $D_{y,i-1}$ and $D_{y,i}$ are respectively control points coordinates and first derivative value (at the beginning and at the end). From equations (4) e (5) it is possible to obtain algebraic expressions of coefficients:

(6) $a_{0y,i} = y_{i-1}$
$a_{1y,i} = D_{y,i-1}$
$a_{2y,i} = 3 \cdot (y_i - y_{i-1}) - 2 \cdot D_{y,i-1} - D_{y,i}$
$a_{3y,i} = 2 \cdot (y_{i-1} - y_i) + D_{y,i-1} + D_{y,i}$

C^2 continuity, necessary for road modelling curve, is obtained with the following second derivative conditions in the n -1 internal contact points:

(7) $Y_i''(1) = Y_{i+1}''(0)$, with i = 1, 2, …, n;

substituting in (7) the expressions of second derivative of curve we have:

(8) $2 \cdot a_{2y,i} + 6 \cdot a_{3y,i} = 2 \cdot a_{2y,i+1}$

and for the (6):

(9) $2 \cdot [3 \cdot (y_i - y_{i-1}) - 2 \cdot D_{i-1} - D_i] + 6 \cdot [2 \cdot (y_{i-1} - y_i) + D_{i-1} + D_i] = 2 \cdot [3 \cdot (y_{i+1} - y_i) - 2 \cdot D_i - D_{i+1}]$

simplifying the previous we obtain:

(10) $D_{i-1} + 4 \cdot D_i + D_{i+1} = 3 \cdot (y_{i+1} - y_{i-1})$

We have 4n-2 equations from the (6) and the (10) and two further equations imposing left end and right end conditions for determining algebraic coefficients of polynomial segments: end conditions regard first or second derivative value that characterize tangent and curvature value.
E.g. requiring that the curvatures at the endpoints are zero, we obtain the called *Natural Cubic Spline*. To obtain null curvature at the endpoints we impose that the second derivative at the endpoints is equal to zero. For the beginning of the curve we have:

(11) $2 \cdot a_{2y,1} = 0$

and for the (6):

(12) $2 \cdot [3 \cdot (y_1 - y_0) - 2 \cdot D_{y,0} - D_{y,1}] = 0$

and simplifying:

(13) $2 \cdot D_{y,0} + D_{y,1} = 3 \cdot (y_1 - y_0)$

The (13) for the end of the curve is:

(14) $D_{y,n} + 2 \cdot D_{y,n} = 3 \cdot (y_n - y_{n-1})$

Representing (10), (13) and (14) in matrix form we have:

(15)
$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & & \\ & & & . & . & . & & \\ & & & & . & . & . & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} D_{y,0} \\ D_{y,1} \\ D_{y,2} \\ D_{y,3} \\ . \\ . \\ D_{y,n-1} \\ D_{y,n} \end{bmatrix} = \begin{bmatrix} 3(y_1 - y_0) \\ 3(y_2 - y_0) \\ 3(y_3 - y_1) \\ 3(y_4 - y_2) \\ \\ \\ 3(y_n - y_{n-2}) \\ 3(y_n - y_{n-1}) \end{bmatrix} \Rightarrow [\mathbf{H}] \times [\mathbf{D}] = [\mathbf{Y}] \Rightarrow [\mathbf{D}] = [\mathbf{Y}] \times [\mathbf{H}]^{-1}$$

$[D_y]$ vector can be determined inverting **[H]** matrix and then we can define through the (6) algebraic coefficients of parametric expressions (1) for each polynomial segments of curve. This method is used for each dependent variable x, y and z.

Quintic Hermite Polynomials

In road design we need to define tangent and curvature values (through first and second derivative) at the beginning and at the end of the road alignment. In this paper, for this necessity, we considered Hermite polynomials with an order greather than 3. In particular, we considered quintic Hermite polynomials to obtain jerk continuity (we need at least C^3 continuity) and to have four conditions (two for each curve extreme) to fix tangent and curvature values for the beginning and the end of curve.
Parametric equations $Q_i(u)$ for each segment of curve are:

(16)
$$\begin{aligned} \mathrm{x_i}(u) &= a_{5x,i}u^5 + a_{4x,i}u^4 + a_{3x,i}u^3 + a_{2x,i}u^2 + a_{1x,i}u + a_{0x,i} \\ \mathrm{y_i}(u) &= a_{5y,i}u^5 + a_{4y,i}u^4 + a_{3y,i}u^3 + a_{2y,i}u^2 + a_{1y,i}u + a_{0y,i} \\ \mathrm{z_i}(u) &= a_{5z,i}u^5 + a_{4z,i}u^4 + a_{3z,i}u^3 + a_{2z,i}u^2 + a_{1z,i}u + a_{0z,i} \end{aligned}$$

with $u \in [0; 1]$ and i=1,2,…,n.
With n+1 control points, number of equations requested for problem solution is 6·3·n, i.e. 6n coefficients for each (x-u, y-u, z-u) plane.
The available equation number (for each parametric plane) is:

- 2·n equations for segment passing through control points;
- (n-1) equations for first derivative continuity in contact points (C^1 continuity);
- (n-1) equations for second derivative continuity in contact points (C^2 continuity);
- (n-1) equations for third derivative continuity in contact points (C^3 continuity);
- (n-1) equations for fourth derivative continuity in internal points (C^4 continuity).

In this case the total equation count is:

(17) 6n-4

while we need of 6n equation.
The other four equations may be obtained, also in this case, setting tangent and curvature value at the beginning and at the end of curve.
In matrix form (18), we have:

(18)
$$\begin{bmatrix} 1 & & & & & & & & & & & & & & & & \\ & & & & & & & & 1 & & & & & & & & \\ -24 & & 24 & & & & & & -3 & 18 & -3 & & & & & \\ & -24 & & 24 & & & & & & -3 & 18 & -3 & & & & \\ & & . & & . & & & & & & . & & . & & & \\ & & & -24 & & 24 & & & & & & -3 & 18 & -3 & & \\ & & & & -24 & & 24 & & & & & & -3 & 18 & -3 & \\ -168 & -384 & -168 & & & & & & -24 & & 24 & & & & & \\ & -168 & -384 & -168 & & & & & & -24 & & 24 & & & & \\ & & . & & . & & & & & & . & & . & & & \\ & & & -168 & -384 & -168 & & & & & & -24 & & 24 & & \\ & & & & -168 & -384 & -168 & & & & & & -24 & & 24 \\ & & & & & & 1 & & & & & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ . \\ D_{m-1} \\ D_m \\ D''_0 \\ D''_1 \\ D''_2 \\ D''_3 \\ . \\ D''_{m-1} \\ D''_m \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 60 \cdot (y_0 - 2 \cdot y_1 + y_2) \\ 60 \cdot (y_1 - 2 \cdot y_2 + y_3) \\ . \\ 60 \cdot (y_{m-3} - 2 \cdot y_{m-2} + y_{m-1}) \\ 60 \cdot (y_{m-2} - 2 \cdot y_{m-1} + y_m) \\ 360 \cdot (y_0 - y_2) \\ 360 \cdot (y_1 - y_3) \\ . \\ 360 \cdot (y_{m-3} - y_{m-1}) \\ 360 \cdot (y_{m-2} - y_m) \\ k_3 \\ k_4 \end{bmatrix}$$

i.e. **[H]** x **[D]** = **[Y]** and **[D]** = **[Y]** x $[\mathbf{H}]^{-1}$.
The parameters k_1, k_2, k_3 e k_4 are first and second derivative values at the beginning and at the end of curve:

(19) $D^{(1)}_0 = k_1$
$D^{(2)}_0 = k_2$

$D^{(1)}_{n-1} = k_3$
$D^{(2)}_{n-1} = k_4$

Algebraic coefficients of parametric expressions (17), are determined with the same method used for Cubic Hermite Polynomials.

Composite Bezier curves of 4° and 6° degree

Bezier's curves (Mortenson, 1985) are polynomials that only approximate given set of n+1 points and are constrained to pass only through the first and the last point of series. Any point on a curve segment is given by a parametric function as following:

$$P(u) = \sum_{i=0}^{n} P_i \cdot B_{i,n}(u) \qquad \text{with } i=0,1,2,\ldots,n; \tag{20}$$

where P(u) = P[x(u) y(u) z(u)] is position vector of any curve's point when *u* change, P_i represent n+1 vertices of a characteristic polygon and the equations:

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} \cdot u^i \cdot (1-u)^{n-i} \tag{21}$$

are a family of function called *Bernstein's functions* that generate and characterize the curve.
Expanding (21) equation for each Cartesian coordinate we have:

$$\begin{aligned} x(u) &= \sum_{i=0}^{n} x_i \cdot B_{i,n}(u) \\ y(u) &= \sum_{i=0}^{n} y_i \cdot B_{i,n}(u) \\ z(u) &= \sum_{i=0}^{n} z_i \cdot B_{i,n}(u) \end{aligned} \tag{22}$$

We can consider a matrix form of Bezier's curves. E.g. for n=5 (characteristic polygon with 5 points, 4° degree curve) we have:

$$P(u) = U_4 \cdot M_4 \cdot B_4 \tag{23}$$

where:

$$U_4 = \begin{bmatrix} u^4 & u^3 & u^2 & u & 1 \end{bmatrix} \tag{24}$$

$$M_4 = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & -12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \end{bmatrix}$$

The previous parametric functions show that the Bezier curve order depends on the number of points of the characteristic polygon. Since in road modelling we have a great number of control points, to limit curve's degree we will compose a curve with polynomial segment with fixed degree, taking care to have continuity in the contact points.
In fact, if the curve must have C^k continuity, we will join polynomial segment with k+1 degree; besides for obtaining C^k continuity in the contact points we will align k control points for each side of polynomial segment. E.g. for having C^2 continuity curve (like actually used for road design) we will align 3 control points for each extreme of polynomial segment, thus each polynomial segment will be generated with 5 control points having 4°degree like shows Figure 2. We will align 4 control points for each internal point for having C^3 continuity (to control jerk development) obtaining a curve generated with 7 control points (6° degree, Figure 3).

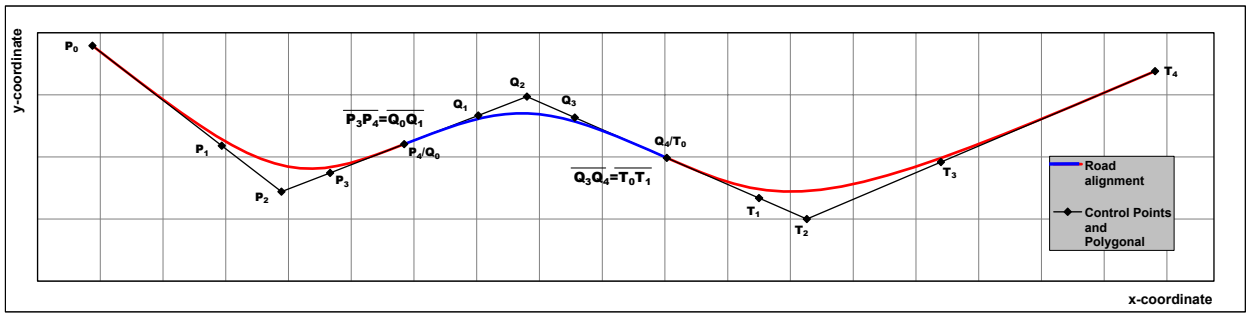


Figure 2 – 4° degree Composite Bézier Curves Planimetry

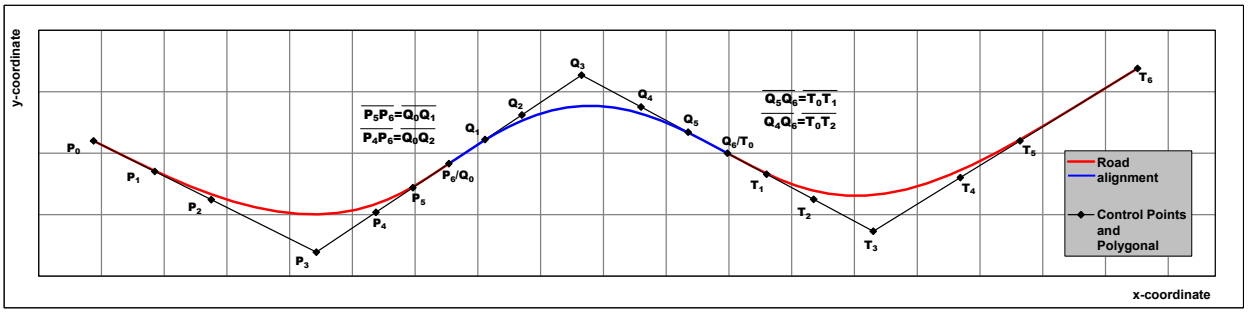


Figure 3 - 6° degree Composite Bézier Curves Planimetry

Matrix form for Bézier's curves of 6° degree is:

$$P(u) = U_6 \cdot M_6 \cdot B_6 \tag{25}$$

where:

$$U_6 = \begin{bmatrix} u^6 & u^5 & u^4 & u^3 & u^2 & u & 1 \end{bmatrix} \tag{26}$$

$$B_6 = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 15 & -60 & 90 & -60 & 15 & 0 & 0 \\ -20 & 60 & -60 & 20 & 0 & 0 & 0 \\ 15 & -20 & 15 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we use the same method to control the tangent and curvature of the endpoints seen for the 4^{th} order Bezier curves. Finally to have a curve with C^3 continuity in all contact point we will align four control points for each afferent polynomial segment to the considered control point.

B-Spline Curves

B-Spline curves (Mortenson, 1985), like Bezier curves, are curves that approximate and approach given set of n+1 points and pass only through the first and the last point of series like shows Figure 4.

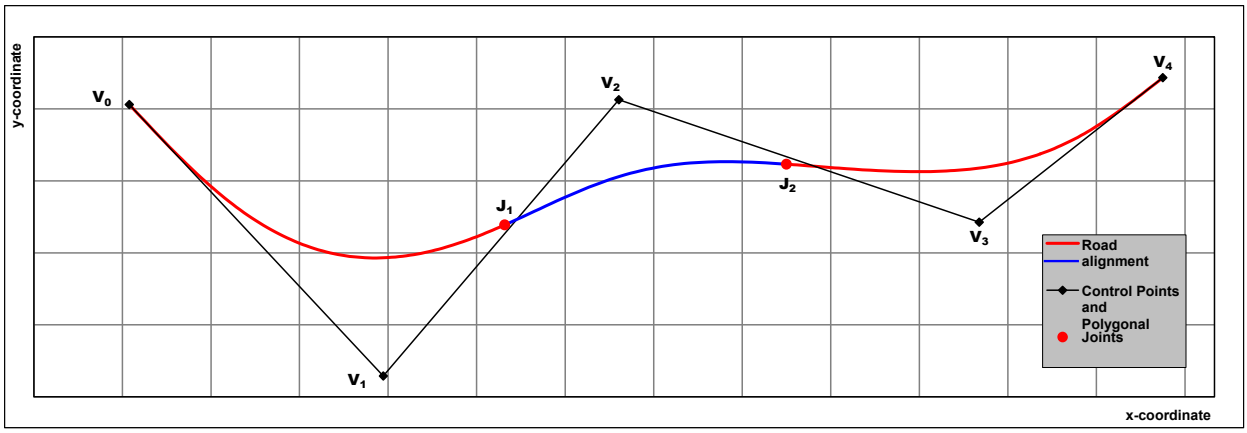


Figure 4 – B-Spline Curves Planimetry

Parametric expression of these curves is similar to the mathematical formulation of Bezier curves but with different blending functions ($N_{i,k}(u)$, in this case):

(27) $P(u) = \sum_i P_i \cdot N_i(u)$ with $u \in [0; n-k+2]$, k=curve's order (to set opportunely);

whence:

(28)
$$x(u) = \sum_i x_i \cdot N_{i,k}(u)$$
$$y(u) = \sum_i y_i \cdot N_{i,k}(u)$$
$$z(u) = \sum_i z_i \cdot N_{i,k}(u)$$

B-Spline's blending functions have the following properties:

1. allow the independence between polynomial segment order and number of control points;
2. permit the local control of the shape that is the displacement of one control point position cause only a part of curve alteration;

in the definition of B-Spline curves are important:

- k curve's order that assign k-1 degree and C^{k-2} continuity to the constituent polynomial segment;
- recursively blending functions $N_{i,k}(u)$ determined as to follow.

For blending functions definition we make some preliminaries. Contact points of polynomial segment are called *joints*. The parameter of blending functions (t_i) is called *knot value* and relate the parametric variable *u* to the control points. The knot values vector (t_0, … , t_{n-k+2}) is defined as follow:

(29)
$$t_i = 0 \quad \text{if } i < k$$
$$t_i = i-k+1 \quad \text{if } k \le i \le n$$
$$t_i = n-k+2 \quad \text{if } i > n$$

In particular we have a *uniform* curve if knots distance are equal.
So blending functions $N_{i,k}(u)$ are defined by the following expressions:

(30) for k=1: $$N_{i,1}(u) = \begin{cases} 1 & \text{if } t_i \le u \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

(31) for k>1: $$N_{i,k}(u) = \frac{(u - t_i) \cdot N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u) \cdot N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}}$$, with the convention (0/0) = 0.

When k value increase curve approximate less and less the characteristic polygon: in fact for k=1 the curve degenerate in the control points, for k=2 correspond with the characteristic polygon, for k>1 we have a curve that approach the characteristic polygon and finally for k=∞ coincide with the line that join the first and the last control point.
In road design we need at least the continuity of curvature (C^2 continuity); following that usable Bezier curves must have k order equal or higher than 4 with the degree of polynomial segment equal or greater than 3 and continuity equal or higher than C^2.
As already seen we need to set tangent and curvature values at the beginning and at the end of curve. For these curves the tangent direction is fixed by first and second or last but one and last control point. Initial and final curvature value is determined by relative position of first three or last three control points. In particular aligned points give null value of curvature, while to have a wanted value of curvature we will must define a suitable relative position of first or last three control points.

N.U.R.B.S. Curves (Non Uniform Rational B-Spline)

The use of N.U.R.B.S. curves (Mortenson, 1985) allow to have a better local control of curve than B-Spline curves. In fact these curves have a "weight" function for each control points that represent numerically the "attraction" for curve by control point. B-Spline are particular N.U.R.B.S. where "weight" functions have one value.
These curves have the following form:

$$P(u) = \begin{cases} \sum_i N_{i,k}(u) \cdot X_i \cdot w_i \\ \sum_i N_{i,k}(u) \cdot Y_i \cdot w_i \\ \sum_i N_{i,k}(u) \cdot Z_i \cdot w_i \\ \sum_i N_{i,k}(u) \cdot w_i \end{cases} \tag{32}$$

where w_i are the control points "weights" and usually they have positive value. Eq. (32) can be expressed normalizing the first three equations with the last.

$$P(u) = \sum_{i=0}^{n} P_i \cdot \left[\frac{w_i \cdot N_{i,k}(u)}{\sum_{i=0}^{n} w_i \cdot N_{i,k}(u)} \right] = \sum_{i=0}^{n} P_i \cdot R_i(u), \tag{33}$$

where:

$$R_i(u) = \frac{w_i \cdot N_{i,k}(u)}{\sum_{i=0}^{n} w_i \cdot N_{i,k}(u)} \tag{34}$$

the shape control of curve through "weight" functions value allows to model the curve without move the control points position: this is shown in Figure 5, where the change of vertex "weight" modify curve tension around control point.

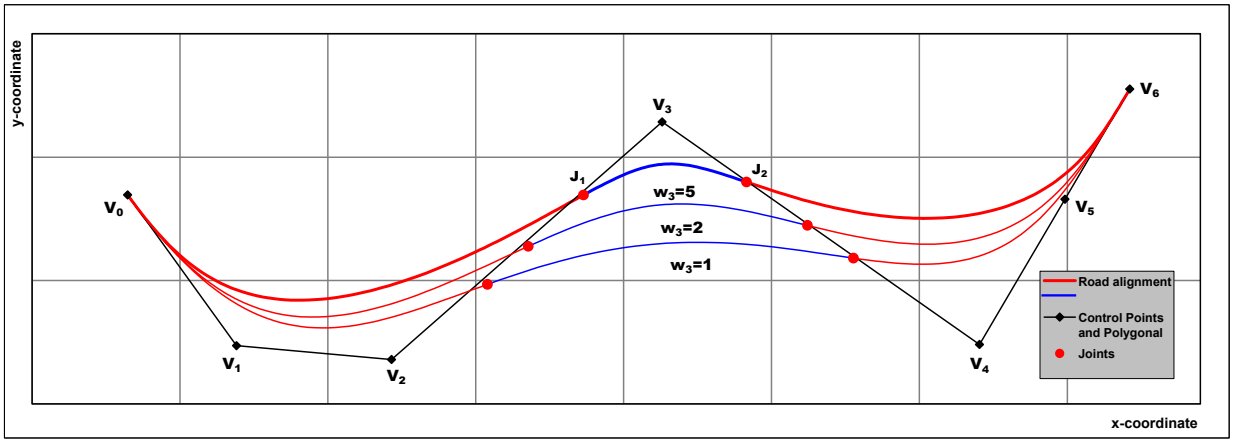


Figure 5 – N.U.R.B.S. Curves Planimetry

ROAD ALIGNMENT MODELLING AND ASSIMILATION TO TRADITIONAL DESIGN

Some characteristics of considered polynomial curves, in particular continuity and degree of freedom, make them suitable for road design, because it is possible to obtain a unitary and continuous geometric definition of road alignment, directly in three-dimensional space. Nevertheless it is to be considered the use of polynomial curves regarding the new Road Design Italian Standards, showing possible advantages or new problems in comparison to traditional Road Design.
New Road Design Italian Standards defined two kind of road geometry constraints:

a) Instructions for single geometric elements (i.e.: curvature value, maximum longitudinal slope, etc.);
b) Instructions for road alignment harmony, planimetric and altimetric coordination and user right perception of road.

For the first kind of constraints, all the necessary verifications can be done with the assimilation procedures considered in this paper.
For the second kind of constrains we have to distinguish road alignment harmony aspects (and the coordination of planimetry and altimetry) from problems of human perception and road-user interaction. For the first problems, characteristics of polynomial curves lead different design approach in comparison

with traditional design method: in fact being polynomial curves continuous and unitary, harmony and spatial coordination are generally obtained, but for further verifications new control indicators (of the road alignment global quality) can be defined, or well-known verifications, not considered in Italian Standards, can be carried out (for example: CCR). For road perception problems, the instructions of Road Design Italian Standards in general are not applicable, because of fundamental difference between polynomial curves and traditional lines (i.e. minimum extension of linear and circular lines). Therefore only visual simulation procedures can give a valid solution to this problem, since such procedures are the final objective of the current research and whose preliminary results are proposed in this paper.

End conditions and degree of freedom

The curves used in road design must allow setting tangent and curvature values at the vertices of curve to fulfil design constraints. The considered polynomial curves are different in such sense because in some cases we can set analytically tangent and curvature of vertices, while in other cases we can do this indirectly through suitable positioning of control points. To understand this better, some properties of such curves are explained below:

- Cubic Hermite curves have only two degree of freedom (for each variable) thus it's possible to set only tangent value or curvature value for each vertex;
- Quintic Hermite curves have four degree of freedom (for each variable) thus it's possible to set analytically tangent value and curvature value for each vertex;
- For composite Bezier curves and for B-Spline and N.U.R.B.S. curves, definition of tangent and curvature vertices value is made with indirect method through the positioning of control points. In fact for these curves initial and final tangents are determined by the first and the last side of characteristic polygon; curvature values of vertices depend on the distance of the second or last but one control point from the direction defined respectively by first and third control point and by last but two and last control point. Figure 6 shows as non zero value of such distance affects vertex curvature value.

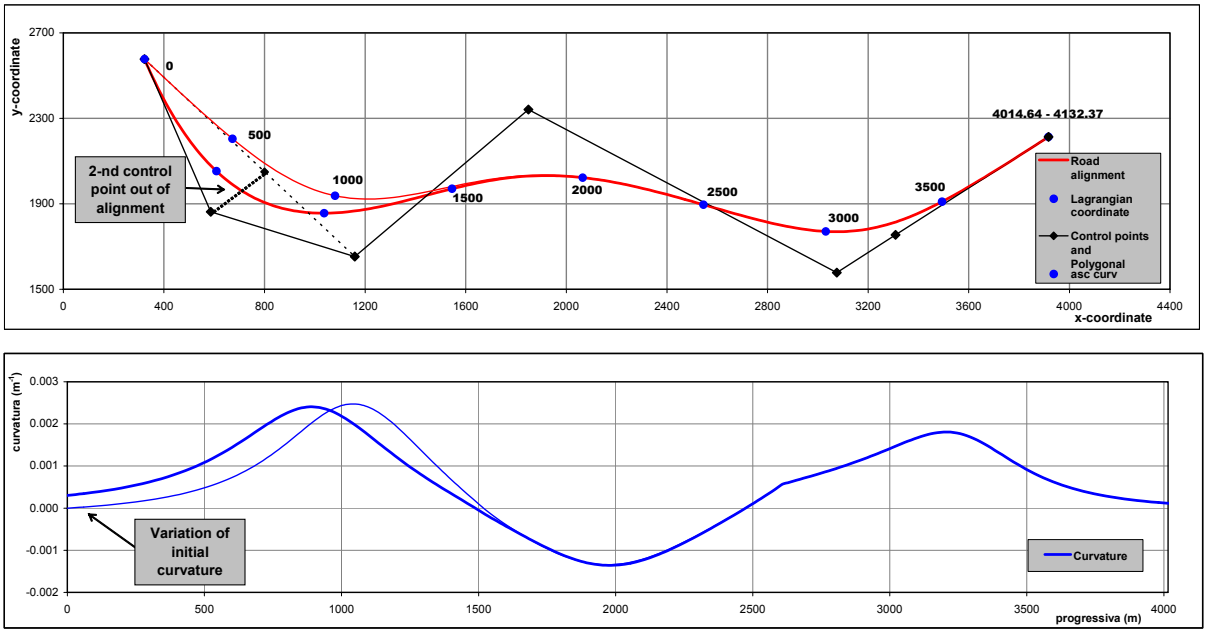


Figure 6 – Control of vertices curvature

All valuations made show that all the typologies of curves seen allow to control, more or less easily, tangent and curvature values of vertices. Control of slope and curvature in each point of road alignment, through geometric characteristic, is important in road design because such control is related with the appropriate kinematics and dynamics requirements that must have theoretic path of vehicles.

Road alignments and kinematics and dynamic requirements

Curve-defining techniques that interpolate or approximate a given set of points are different in geometric modelling of design road alignment because in the first technique control point and curve displacements are equal, while in the second displacements are different for blending functions action.

In traditional road design geometric parameters of curves, that allow the control of kinematic and dynamic conditions, are set before and derive by the theory of motion on a path and by the constraints of the standards. In road design with polynomial curves the verification of geometric parameters follows the road alignment definition.

Cubic and quintic Hermite curve interpolate a given set of points; such curves are usable for road design because have C^2 continuity at least. Figure 7 shows curvature and jerk values along the curve and we can see that the jerk of cubic curves is discontinuous while for the quintic curves this is continuous.

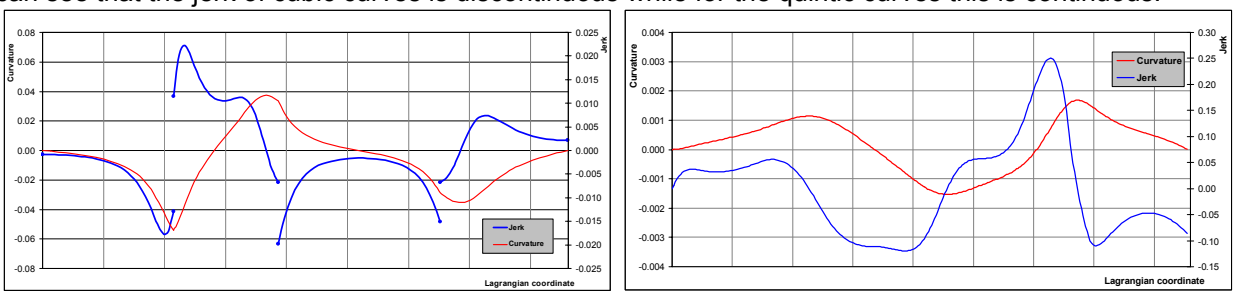


Figure 7 – Curvature and Jerk in cubic and quintic Hermite curves

A control point movement in Hermite curves propagate its effect along whole curve (see Figure 8); this is a variation diminishing shifting from control point considered.

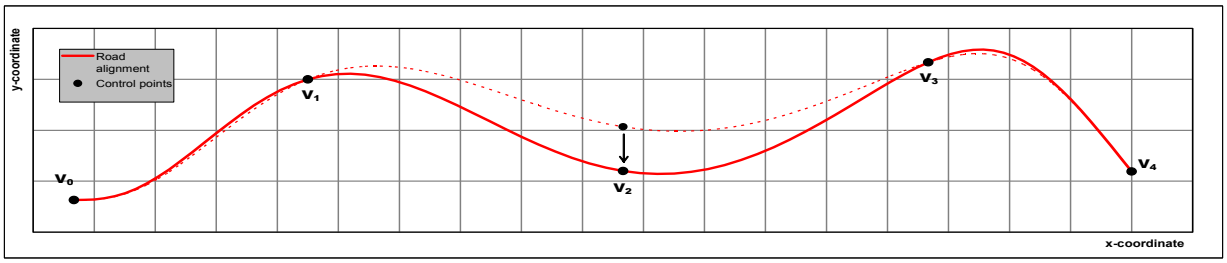


Figure 8 – Effects of control point movement in Hermite Curves

Use of composite Bezier curves is not easy because need to control degree of constituent polynomial segments. Besides final and initial control points of adjoining segments must be aligned and symmetrical regards contact point; number of aligned points depend by wanted continuity. For example to obtain C^3 continuity curve four control points for each adjoining segment must be aligned for every contact point; thus every part of curve is generated with seven control points (Figure 3) getting 6° degree polynomial segment. Figure 9 shows an example of road alignment modelling using Bezier curve control of curvature value at the beginning of the curve.
These curves verify essential characteristic of use because have C^3 continuity at least, obtaining curvature and jerk continuity along whole road alignment.

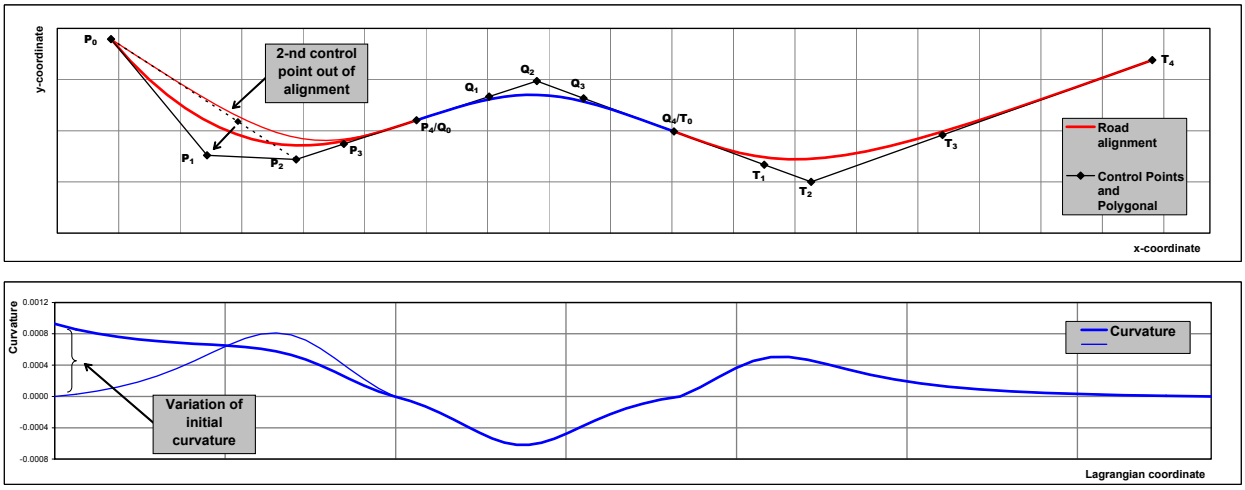


Figure 9 – Control of vertices curvature

In Bézier curves two aligned side of characteristic polygon give a segment with zero curvature value, maintaining unified definition of road alignment. Furthermore shape of curve can be modified through coincident control points (two and more) increasing control point attraction towards the curve.
Use of other curves that approximate given set of points, as B-Spline and N.U.R.B.S., is enough easy because allows local control of curve and then only a limited part of curve is modified by control point movement. In particular, N.U.R.B.S. curves are particularly effective because through “weight” of control points permit to change attraction of control point towards the curve; this property allows to modify road alignment without to move control points (rif. Figure 5).

Regards control of kinematics ad dynamic parameters, 4° order curves are C^2 continuity and 5° order curves are C3 continuity: curvature is continuous in both cases and in the second one also jerk is continuous.
Besides for B-Spline and N.U.R.B.S. curves some collinear vertices give segment with zero curvature and coincident control points increase attraction of multiple control point towards the curve. Figure 10 shows an example of B-Spline road alignment modelling with co-linear vertices.

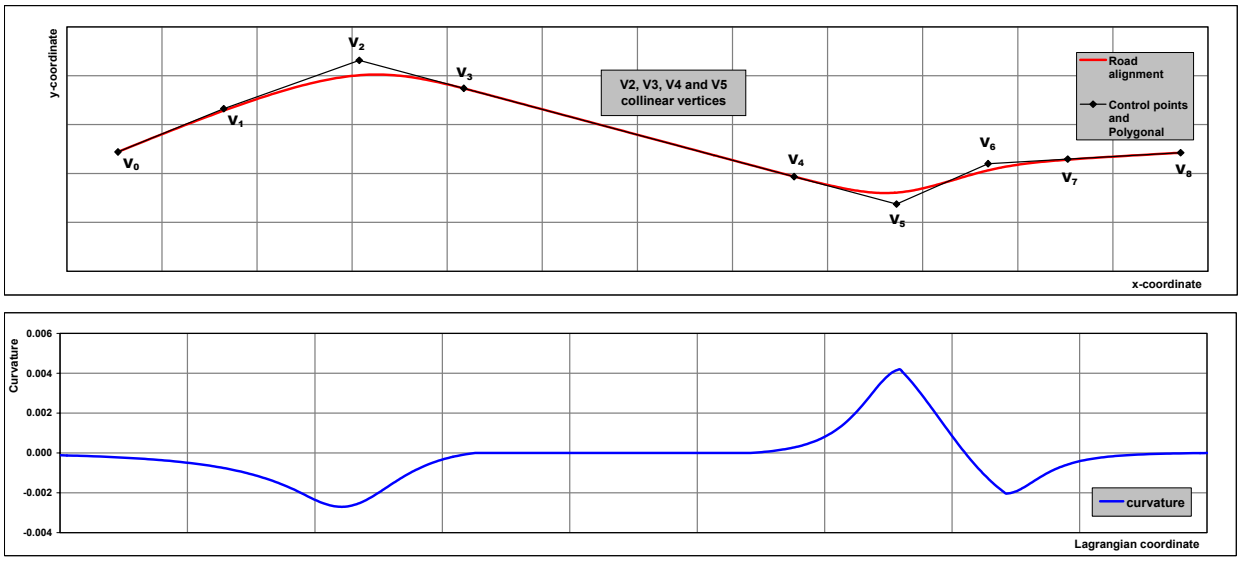


Figure 10 – Collinear vertices in B-Spline modelling

Traditional road alignment assimilation

The new Italian Road Design Standard (*"Norme funzionali e geometriche per la costruzione delle strade", D.M. 5.11.2001*) allow the use of polynomial curves for the road alignment design, but impose verification as for the traditional road alignment. Numeric procedures to develop such verifications treating polynomial curves as traditional are not defined.
In this paper is shown a possible procedure for making these verifications. Such procedure define road alignment through Lagrange's coordinate *s* (curvilinear coordinate), and express through s characteristic greatness (planimetric an altimetric curvature, jerk, longitudinal slope and so on).

Assimilation of planimetric road alignment
Assimilation criterion is based on comparison of the dynamic variables referred to the material point motion along a given path defined by the designed road alignment. In particular, relations to determine planimetric curvature and jerk come from derivative of parametric functions x(u), y(u) e z(u), considering composite functions properties.
Expression (35) can be considered for the analysis and the determination of planimetric curvature and jerk (variation of transversal acceleration in the time) along road alignment, comes from derivatives of parametric functions. As for traditional road alignment, the considered planimetric curvature is the same of crooked curve projection on x-y plane: the approximation is allowed because longitudinal slope is limited.

$$(35)\quad \frac{1}{R} = \frac{d^2y/dx^2}{\left[1+\left(dy/dx\right)^2\right]^{3/2}} = \frac{\frac{dx}{du}\cdot\frac{d^2y}{du^2} - \frac{d^2x}{du^2}\cdot\frac{dy}{du}}{\left[\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2\right]^{3/2}}$$

jerk values distribution along road alignment, constant speed setting, is expressed as following:

$$(36)\quad C = \frac{da_c}{dt} = \frac{d\left(\frac{v^2}{R}\right)}{d\left(\frac{s}{v}\right)} = v^3\cdot\frac{d\left(\frac{1}{R}\right)}{ds}$$

curvature derivative regards curvilinear coordinate is:

$$(37)\ \frac{d\left(\frac{1}{R}\right)}{ds}=\frac{d\left(\frac{1}{R}\right)\Big/dx}{ds\Big/dx}$$

deriving expression (35) regards *x* variable we have:

$$(38)\ \frac{d\left(\frac{1}{R}\right)}{dx}=\frac{d}{dx}\left\{\frac{d^2y\Big/dx^2}{\left[1+\left(dy\Big/dx\right)^2\right]^{3/2}}\right\}=\frac{d^3y}{dx^3}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-3/2}+\frac{d^2y}{dx^2}\cdot\left\{-\frac{3}{2}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-5/2}\cdot 2\cdot\left(dy\Big/dx\right)\cdot\frac{d^2y}{dx^2}\right\}$$

furthermore:

$$(39)\ \frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

substituting (38) and (39) in the (37) we obtain:

$$(40)\ \frac{d\left(\frac{1}{R}\right)}{ds}=\frac{d^3y}{dx^3}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-2}+\frac{d^2y}{dx^2}\cdot\left\{-\frac{3}{2}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-3}\cdot 2\cdot\left(dy\Big/dx\right)\cdot\frac{d^2y}{dx^2}\right\}$$

and finally we obtain jerk expression:

$$(41)\ C=v^3\cdot\frac{d\left(\frac{1}{R}\right)}{ds}=v^3\cdot\left\{\frac{d^3y}{dx^3}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-2}+\frac{d^2y}{dx^2}\cdot\left\{-\frac{3}{2}\cdot\left[1+\left(dy\Big/dx\right)^2\right]^{-3}\cdot 2\cdot\left(dy\Big/dx\right)\cdot\frac{d^2y}{dx^2}\right\}\right\}$$

For using jerk expression y derivatives regards x, given by (42),(43) and (44), must be substituted in the expression (41). In this way jerk values can be determined for each curvilinear coordinate value to make expected verifications in comparison with fixed limitations of traditional road alignment.

$$(42)\ \frac{dy}{dx}=\frac{dy\Big/du}{dx\Big/du}$$

$$(43)\ \frac{d^2y}{dx^2}=\frac{\frac{d^2y}{du^2}-\left(\frac{dy\Big/du}{dx\Big/du}\right)\cdot\frac{d^2x}{du^2}}{\left(\frac{dx}{du}\right)^2}$$

$$(44)\ \frac{d^3y}{dx^3}=\frac{\frac{d^3y}{du^3}-\left[\frac{\frac{d^2y}{du^2}-\left(\frac{dy\Big/du}{dx\Big/du}\right)\cdot\frac{d^2x}{du^2}}{\left(\frac{dx}{du}\right)^2}\right]\cdot\left(2\cdot\frac{dx}{du}\cdot\frac{d^2x}{du^2}\right)-\left\{\left[\frac{\frac{d^2y}{du^2}-\left(\frac{dy\Big/du}{dx\Big/du}\right)\cdot\frac{d^2x}{du^2}}{\left(\frac{dx}{du}\right)^2}\right]\cdot\frac{dx}{du}\right\}\cdot\frac{d^2x}{du^2}-\left(\frac{dy\Big/du}{dx\Big/du}\right)\cdot\frac{d^3x}{du^3}}{\left(\frac{dx}{du}\right)^3}$$

Assimilation of altimetric road alignment:longitudinal slope and vertical links

We need to get expressions of punctual longitudinal slope and vertical curvature along road alignment, regards parametric derivatives. The function that define altimetric develop give coordinate *z* regards curvilinear coordinate *s* (measured on horizontal plane); longitudinal slope (45) and vertical curvature (46) are defined by first and second derivative of this function.

(45) $$i = \frac{dz}{ds_h}$$

(46) $$\frac{1}{R_v} = \frac{\frac{d^2z}{ds_h^2}}{\left[1+\left(\frac{dz}{ds_h}\right)^2\right]^{1/2}}$$

considering:

(47) $$ds_h = \sqrt{dx^2 + dy^2}\ ,$$

(48) $$\frac{ds_h}{du} = \frac{\sqrt{dx^2 + dy^2}}{du} = \frac{dx}{du}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$ and

(49) $$\frac{d^2s_h}{du^2} = \frac{d}{du}\left[\frac{dx}{du}\cdot\sqrt{1+\left(\frac{dy/du}{dx/du}\right)^2}\right] = \frac{d}{du}\left\{\frac{dx}{du}\cdot\left[1+\left(\frac{dy/du}{dx/du}\right)^2\right]^{1/2}\right\}$$

equations (45) and (46) an be expressed through parametric derivatives regards *u* of x, y e z variables:

(50) $$i = \frac{dz}{ds_h} = \frac{dz/du}{ds_h/du}$$

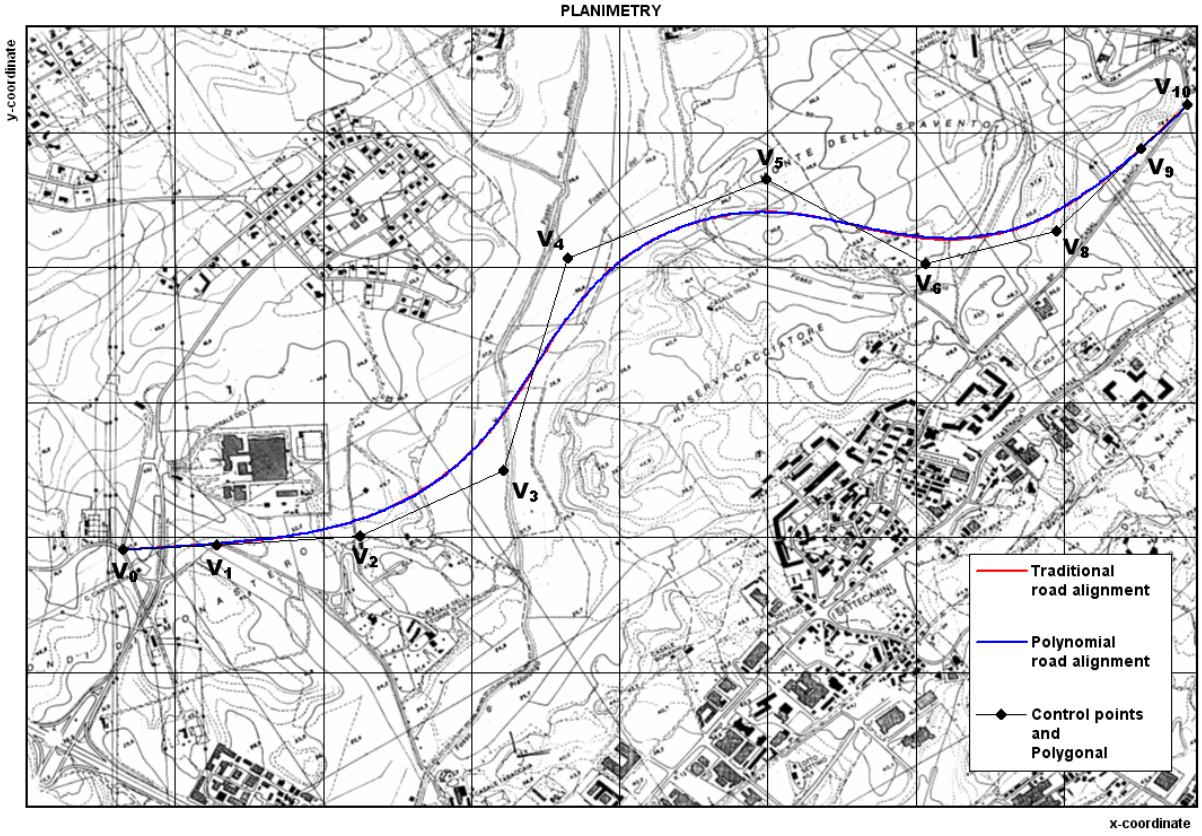
(51) $$\frac{1}{R_v} = \frac{\frac{d^2z}{du^2} - \frac{dz/du}{ds/du}\cdot\frac{d^2s}{du^2}}{\left(ds/du\right)^2}}{\left[1+\left(\frac{dz/du}{ds/du}\right)^2\right]^{1/2}}$$

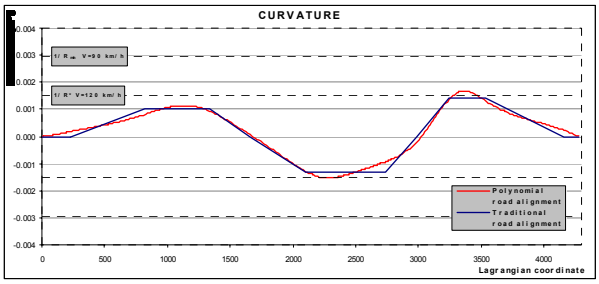
and to determine longitudinal slope and vertical curvature for each curvilinear coordinate value, finally to make expected verifications in comparison with fixed limitations of traditional road alignment

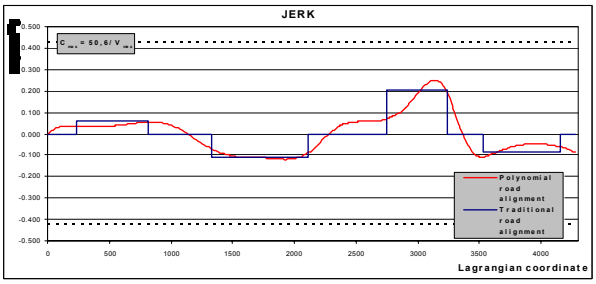
EXAMPLES OF ROAD ALIGMENT DESIGN

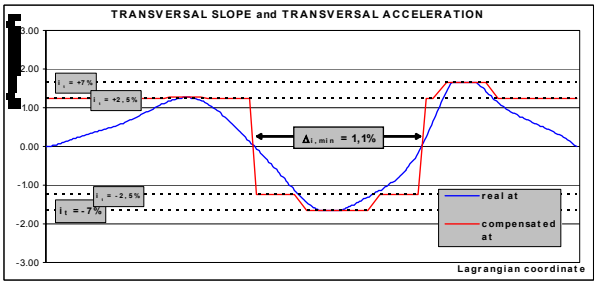
Design of suburban road alignment with one carriageway has been realized to estimate applicability of polynomial curves to road modelling. These examples allow to make a comparison between design of road alignment realized with polynomial curves (Hermite, Bézier e N.U.R.B.S.) and with traditional elements. Such comparison concerns characteristic geometric, kinematic and dynamic parameters (longitudinal and transversal slope, planimetric and altimetric curvature, speed, transversal acceleration, jerk). Results of this comparison are shown in the follow regards each methodology of design.

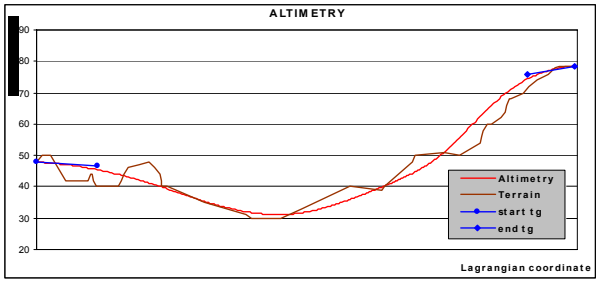
Quintic Hermite curves.

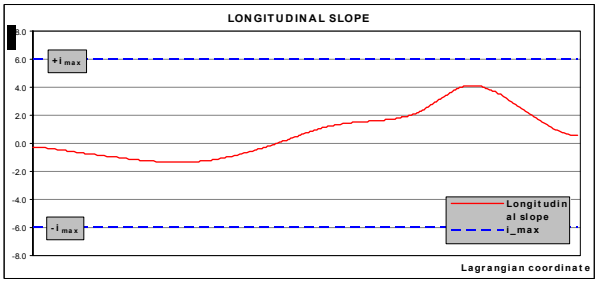


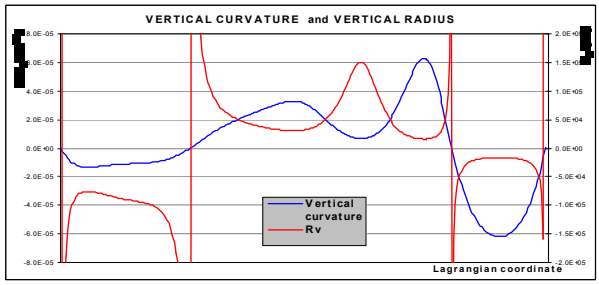




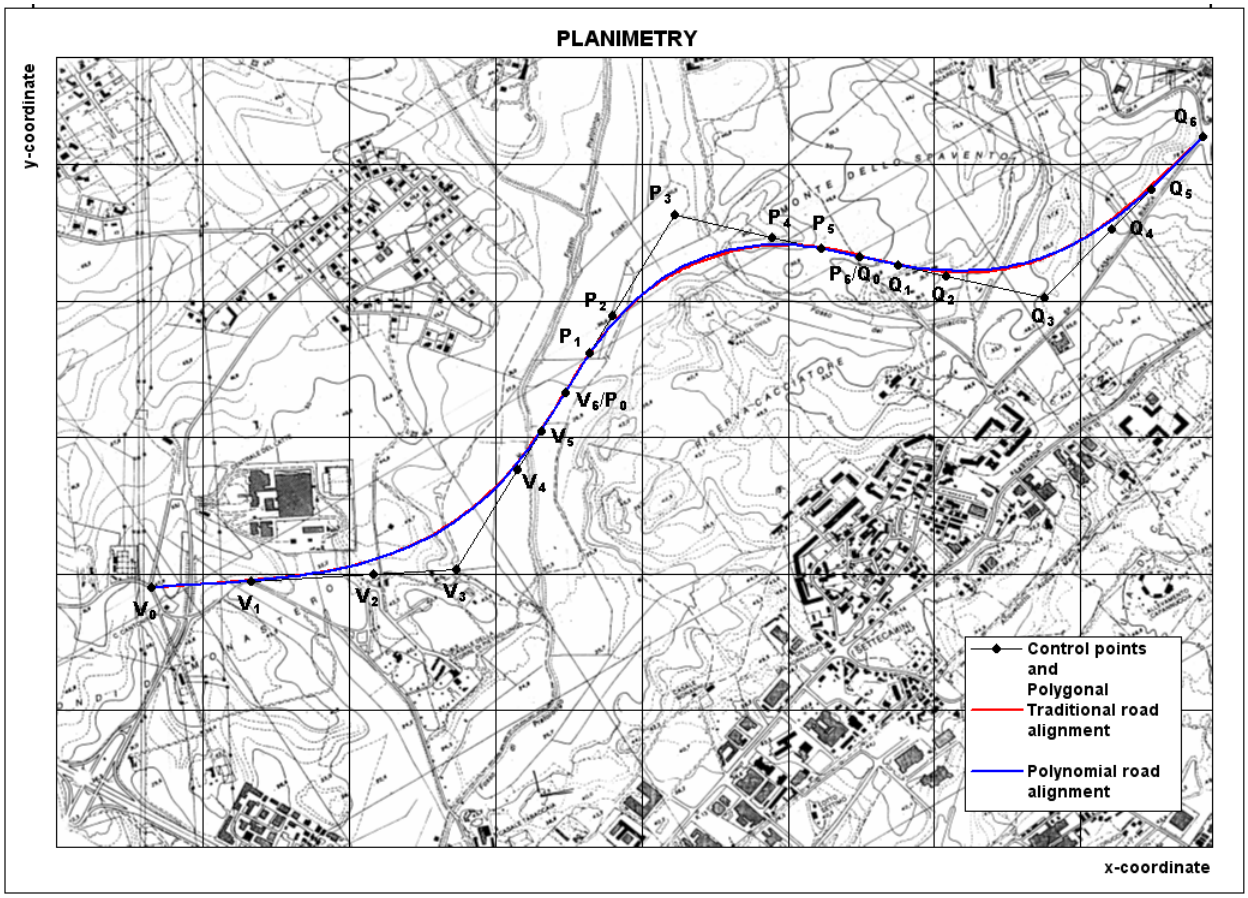


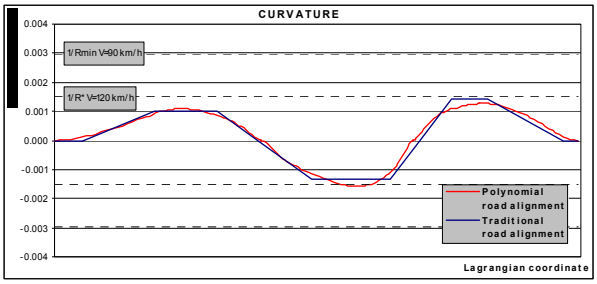


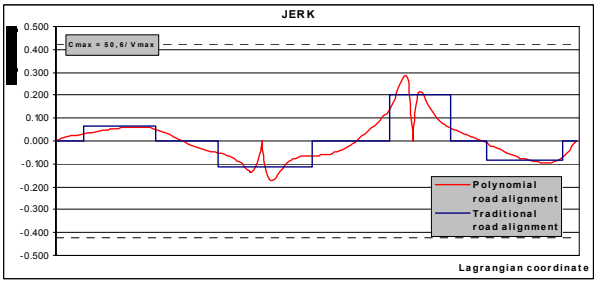


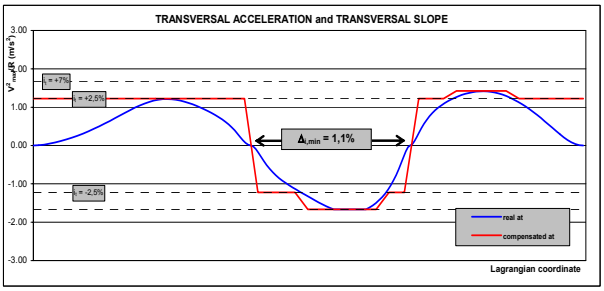


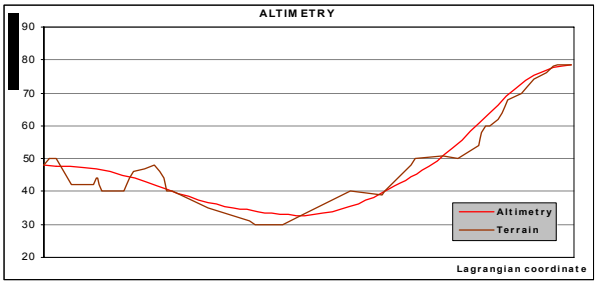
Composite 6° degree Bézier Curves

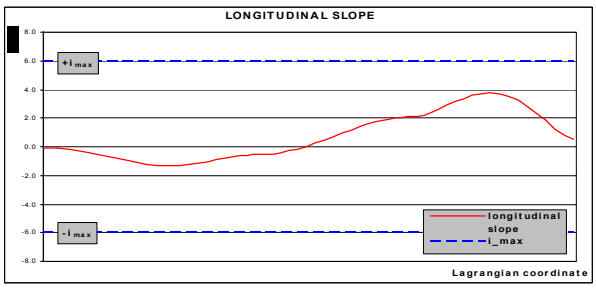


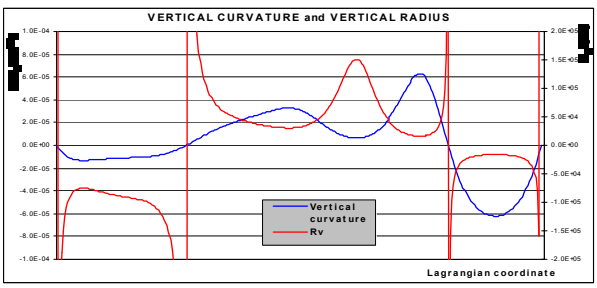




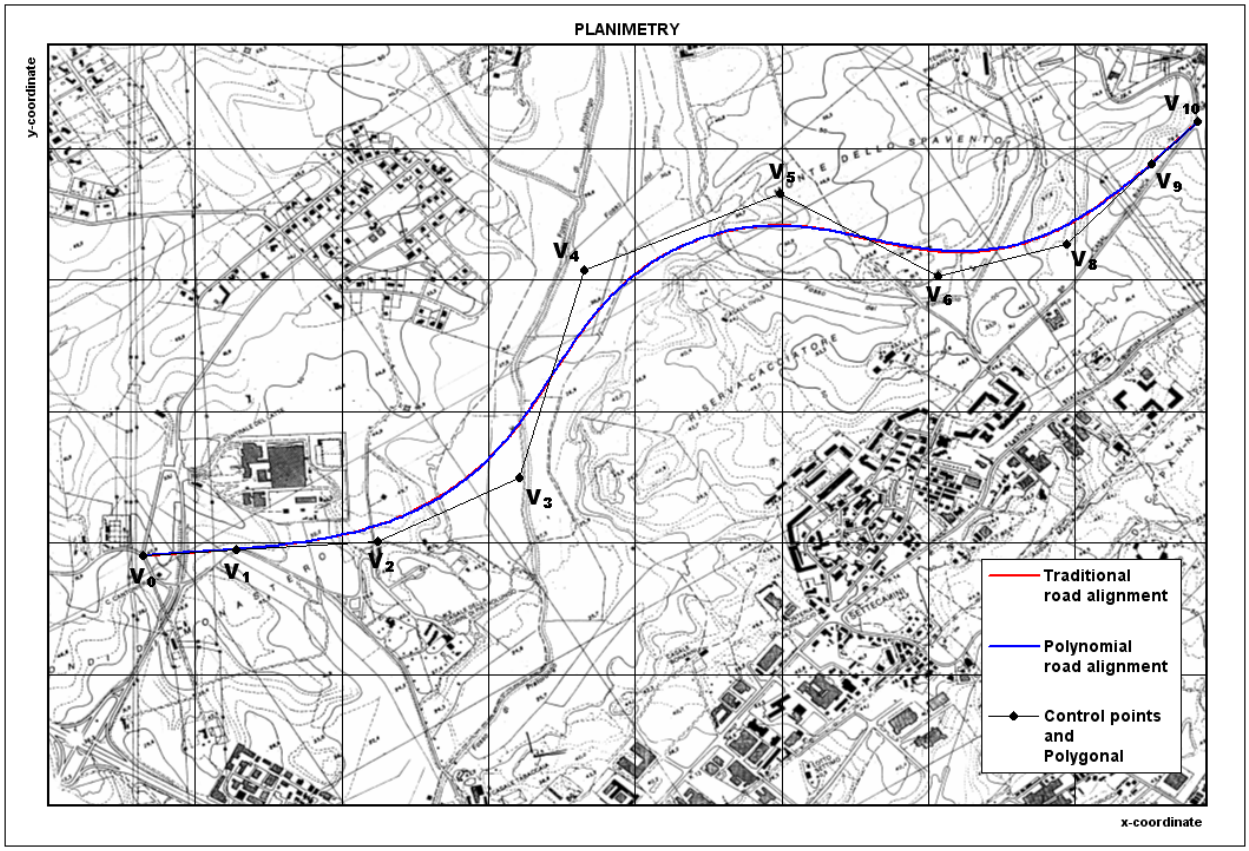


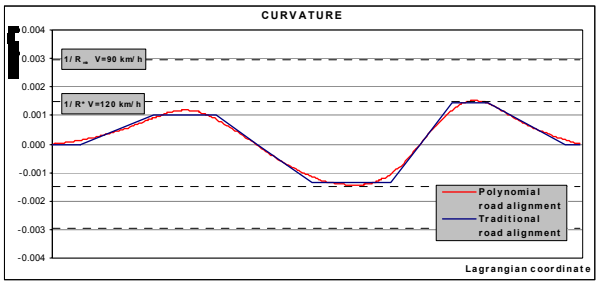


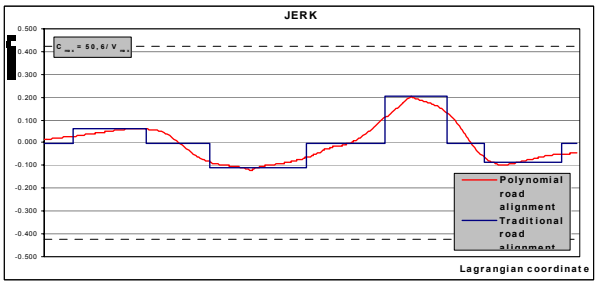


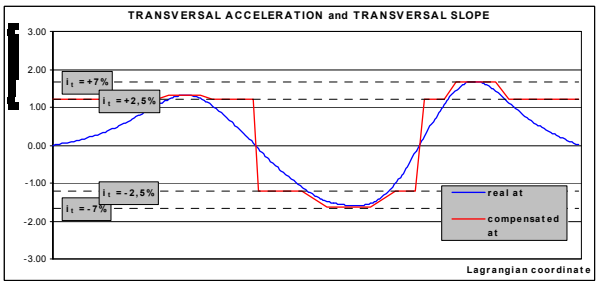


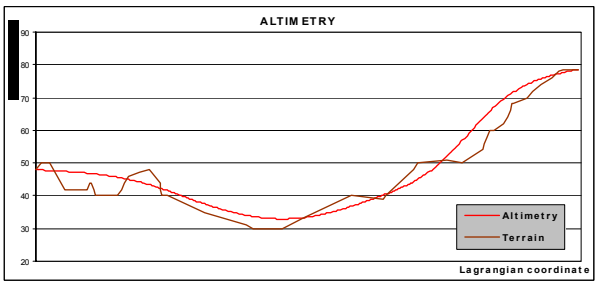
N.U.R.B.S. - k=5

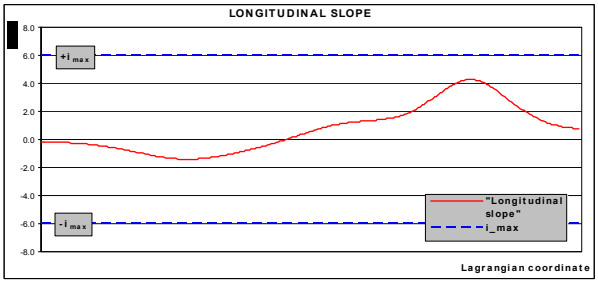

PLANIMETRY
y-coordinate
x-coordinate
V0
V1
V2
V3
V4
V5
V6
V8
V9
V10
Traditional road alignment
Polynomial road alignment
Control points and Polygonal

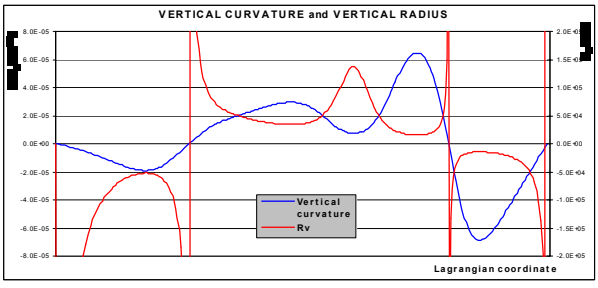

CURVATURE
1/ R_ V=90 km/ h
1/ R* V=120 km/ h
Polynomial road alignment
Traditional road alignment
Lagrangian coordinate


JERK
Polynomial road alignment
Traditional road alignment
Lagrangian coordinate


TRANSVERSAL ACCELERATION and TRANSVERSAL SLOPE
i_t = +7%
i_t = +2,5%
i_t = -2,5%
i_t = -7%
real at
compensated at
Lagrangian coordinate


ALTIMETRY
Altimetry
Terrain
Lagrangian coordinate


LONGITUDINAL SLOPE
+i max
-i max
"Longitudinal slope"
i_max
Lagrangian coordinate


VERTICAL CURVATURE and VERTICAL RADIUS
Vertical curvature
Rv
Lagrangian coordinate

CONCLUSIONS

In this paper the opportunity of using parametric polynomial curves in road design has been considered examining main characteristics of some typology of polynomial curves. Some analytic evaluations have been developed to assimilate such curves to traditional road alignment.
Use of polynomial curves in road design and modelling is an effective tool for solving design problems observing terrain and regulations constraint.
Particularly it was observed that characteristic values (as tangents and curvatures) can be defined at the beginning and at the end of curve, and the road alignment can be modelled in relation to imposed constraints controlling at the same time geometric and dynamic parameters.
Besides assimilation criteria to the traditional road alignment necessary to make prescribed verification are proposed; such criteria are defined in punctual and performance form regards the develop of road alignment.
In relation to the advantages of using polynomial curves in road design, it was observed that traditional road design accepts discontinuity in theoretic distribution of jerk values along the curve, while this problem can be solved through suitable polynomial curve choice (in relation to degree of polynomial). Moreover it has to be specified that traditional road alignment discontinuity of jerk values are not really experienced by road user, because vehicle paths have always continuous variation of curvature due to carriageway width.
Further interest in polynomial road alignment comes from possible use in road guide simulation. In fact, a road alignment modeller, able to use directly such road alignment for road guide simulation, is under advanced development phase in the *Dipartimento di Idraulica Trasporti e Strade* of the *Università di Roma "La Sapienza"*. Main advantage in the use of such polynomial road alignment is unitary numerical definition that allows use in graphic-numerical applications without further elaborations.
Generally increased speeds of road use in new construction and in adjustment of existing roads required more pressing limitations for maximum curvature values and longitudinal slope besides necessity to guarantee continuous and progressive variations of such dynamic and geometric parameters. For the limitations considered in standard road design, road alignment are more constrained and the insertion in the natural environment is less harmonic. In fact, the crossing of not level terrain and presence of particular constraints impose realization of important infrastructures that alter natural terrain order seen as landscape and as geo-morphological equilibrium.
Use of polynomial curves could allow to design road alignment directly in three-dimensional space and consider all extension of alignment as three-dimensional continuous curves in digital terrain models.

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